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LABORATORY OF PLASMA STUDIES
CORNELL UNIVERSITY
ITHACA, NEW YORK

RELATIVISTIC FIELDS DUE TO A PARTICLE IN
A GROUNDED CYLINDRICAL BOX

by

Richard L. Liboff

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ABSTRACT

The exact relativistic fields inside a perfectly conducting, closed, grounded cylindrical box of finite length, which are due to a charged particle moving down the axis in vacuum are obtained. This permits construction of the solution for a line segment propagating down the axis, and the return surface-current. The fields are shown to be causal so that no field is present before the light front. The technique of solution employs superposition of Lorentz transformed fields of the positive and negative images. The initial condition that there be no field in the box at time $t = 0$ is satisfied by adding a superposition of the cavity modes to the inhomogeneous solution. The fields and energy in the wake are also found.

Coulomb field in the frame of the particle to the 'lab' frame.⁵ When boundaries are present such a technique is still possible if the static problem is soluble using the method of images. This idea was suggested by Ott and Shmoys⁶ in an investigation of a particle impinging on a dielectric half space.

The static solution for a particle in a cylindrical box is solved with an infinite array of positive and negative images,⁷ (owing to the infinite number of reflections in the two end plates). This, together with one other observation, permits solution in terms of Lorentz transformations of the positive and negative fields. This second observation is that when the positive source charge (in the box) moves to the right, all of the positive images rigidly move with it while all the negative images rigidly move in the opposite direction. First we calculate the static field of the positive charges and then Lorentz transform this field to the lab (box) frame. Then we calculate the static field of the negative charges (these are all image charges) and Lorentz transform it to the lab frame. Addition of these two sets of fields gives the inhomogeneous part of the solution to the problem. It contains the singularity at the particle position $z = vt$, $r = 0$ and satisfies the boundary condition that the

tangential component of \underline{E} vanishes at the walls. It does not satisfy the initial condition that the fields are zero in the whole box at $t = 0$ (At $t = 0$, the positive and negative images cross. This current generates a magnetic field.) To insert this piece of data we construct a homogeneous solution which cancels the inhomogeneous solution at $t = 0$. The homogeneous solution is a superposition of the TM modes of the cavity. (The inhomogeneous solution is a TM wave).

The addition of the homogeneous solution also guarantees that the whole solution is causal so that all fields vanish for $z > ct$. Furthermore for times $ct < L$ (the box is of length L) the fields are those of a semi infinitely long ($0 \leq z$) cylinder. This must be the case since the fields are not influenced by the far wall before $t = L/c$.

The chief assumption in the analysis is that the particle moves in free flight. This assumption is used in the analysis to obviate a singularity in the energy of the wake fields. For an ultra-relativistic electron it is found that free flight is maintained provided the electron leaves the cavity through a hole of diameter greater than 10^{-13} cm (1 fermi).

3. Analysis

3a. The Inhomogeneous Component

The static potential due to a point charge q on the axis at $z = 0$ interior to an infinitely long grounded cylinder,

with perfectly reflecting walls is,

$$\phi(r, z) = \frac{2q}{r_0^2} \sum_{j=1}^{\infty} \frac{e^{-k_j |z|} J_0(k_j r)}{k_j [J_1(k_j r_0)]^2} \quad (1)$$

The radius of the tube is r_0 . The zeroth and first order Bessel functions are J_0 and J_1 respectively. The above expression contains the proper singularity at $z=0$, through the δ -function representation

$$\delta(r) = \frac{1}{\pi} \sum_j \frac{J_0(k_j r)}{r_0^2 [J_1(k_j r_0)]^2}$$

where

$$\int_0^{r_0} \delta(r) 2\pi r dr = 1 \quad \text{and} \quad J_0(k_j r_0) = 0$$

It follows that

$$2\pi q \delta(r) = \left[\frac{\partial \phi_{>}}{\partial z} - \frac{\partial \phi_{<}}{\partial z} \right]_{z=0}$$

where $>$ denotes $z \geq 0$ respectively.

For a closed cylindrical box with end plates at $z = 0$ and $z = L$ and a test charge q at $z = z_0$, an infinite array of image

charges are induced exterior to the box along the axis. Their relative positions are depicted in Fig. 1. The static solution to this problem is obtained by replacing the exponential z-dependent part of the solution in (1) with the corresponding potential due to all the images. There are positive images at $z_n = 2nL + z_0$ and negative images at $z_n = 2nL - z_0$ where n is an integer running from $-\infty$ to $+\infty$.

For the relativistic problem we must separate the field into that due to the positive images (plus source) and that due to the negative images. The z-dependent part of the solution due to the positive charges appears as,

$$\psi_j^{(+)} = \sum_{n=1}^{\infty} e^{-k_j(2nL - z_0 + z)} + \sum_{n=1}^{\infty} e^{-k_j(2nL + z_0 - z)} + e^{-k_j|z - z_0|} = \frac{\cosh k_j [L - |z - z_0|]}{\sinh k_j L}$$

Similarly the z-dependent part of the solution due to the negative charges appears as,

$$\psi_j^{(-)} = \frac{\cosh k_j [L - |z + z_0|]}{\sinh k_j L}$$

Superposition of the total solutions generated by $\psi^{(+)}$ and $\psi^{(-)}$ (as given by these latter two expressions) reproduces the static solution⁷.

If the test charge is moving on the axis inside the box, all of the positive images move with it. Let the frame where these charges are at rest be $0'$. At $t = 0$ the source charge is at the origin of $0'$ and, furthermore, at this time the $0'$ frame and the lab frame 0 are coincident.

The potential due to all the positive charges in $0'$ (where the source is always at the origin) is,

$$\phi^{(+)}(r', z') = \frac{2q}{r_0} \sum_j \frac{\cosh k_j (L' - |z'|)}{\sinh k_j L'} \frac{J_0(k_j r)}{k_j J_1^2}$$

In this latter formula and expressions follow, J_1^2 is written for $J_1^2(k_j r_0)$.

The corresponding components of the electric field are

$$E_z^{(+)} = \frac{2q}{r_0} \sum_j \frac{\sinh k_j (L' - |z'|)}{\sinh k_j L'} \frac{J_0(k_j r)}{k_j J_1^2} \operatorname{sgn} z'$$

$$E_r^{(+)} = \frac{2q}{r_0} \sum_j \frac{\cosh k_j (L' - |z'|)}{\sinh k_j L'} \frac{J_1(k_j r)}{J_1^2}$$

Next we must write the RHS of the above expressions in terms of the coordinates of the lab frame 0 . The transformation for the positive chain is $z' = \gamma(z - vt)$ and $z' = \gamma(z + vt)$ for the negative chain (see Fig. 2). The (proper) length between image charges in $0'$ is L' while the length between these same charges in 0 is L , so that $L' = \gamma L$. There results,

$$\begin{aligned}
 E_z^{(+)} &= \frac{2q}{r_0^2} \sum_j \frac{\sinh \gamma k_j [L - |z - vt|]}{\sinh \gamma k_j L} \frac{J_0(k_j r)}{k_j J_1^2} \operatorname{sgn}(z-vt) \\
 E_r^{(+)} &= \frac{2q}{r_0^2} \sum_j \frac{\cosh \gamma k_j [L - |z - vt|]}{\sinh \gamma k_j L} \frac{J_1(k_j r)}{J_1^2} \quad (2)
 \end{aligned}$$

$$\gamma \equiv (1 - \beta^2)^{-1/2}, \quad \beta \equiv v/c$$

When observed in the lab frame, these fields become,

$$E_z^{(+)} = E_z^{(+)'}, \quad E_r^{(+)} = \gamma E_r^{(+)'}, \quad B_\phi^{(+)} = \beta E_r^{(+)'}, \quad (3)$$

In similar manner, we obtain for the fields of the negative charges, expressed in the coordinates of 0,

$$\begin{aligned}
 E_z^{(-)} &= \frac{2q}{r_0^2} \sum_j \frac{\sinh \gamma k_j [L - |z + vt|]}{\sinh \gamma k_j L} \frac{J_0(k_j r)}{J_1^2} \operatorname{sgn}(z-vt) \\
 E_r^{(-)} &= \frac{2q}{r_0^2} \sum_j \frac{\cosh \gamma k_j [L - |z - vt|]}{\sinh \gamma k_j L} \frac{J_1(k_j r)}{J_1^2} \quad (4)
 \end{aligned}$$

Transforming these fields to the lab frame gives,

$$E_z^{(-)} = E_z^{(-)'}, \quad E_r^{(-)} = \gamma E_r^{(-)'}, \quad B_\phi^{(-)} = \beta E_r^{(-)'}$$

Superposing these fields with those of the positive chain,

Eq. (3) gives the inhomogeneous component of the total

solution,

$$\begin{aligned}
 E_z^i &= \frac{4q}{r_0^2} \sum_j \frac{J_0(k_j r)}{J_1^2} \frac{\sinh \Omega_j t \cosh K_j (1 - \frac{z}{L})}{\sinh K_j} \\
 E_r^i &= \frac{4q\gamma}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \frac{\sinh \Omega_j t \sinh K_j (1 - \frac{z}{L})}{\sinh K_j} \\
 B_\phi^i &= \frac{4q\gamma\beta}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \frac{\cosh \Omega_j t \cosh K_j (1 - \frac{z}{L})}{\sinh K_j}
 \end{aligned} \tag{6a}$$

for $z > vt$. (The superscript "i" denotes inhomogeneous.)

"Behind" the particle for $z < vt$ these fields become,

$$\begin{aligned}
 E_z^i &= -\frac{4q}{r_0^2} \sum_j \frac{J_0(k_j r)}{J_1^2} \frac{\sinh K_j \frac{z}{L} \sinh (K_j - \Omega_j t)}{\sinh K_j} \\
 E_r^i &= \frac{4q\gamma}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \frac{\sinh K_j \frac{z}{L} \sinh (K_j - \Omega_j t)}{\sinh K_j} \\
 B_\phi^i &= \frac{4q\gamma\beta}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \frac{\cosh K_j \frac{z}{L} \cosh (K_j - \Omega_j t)}{\sinh K_j}
 \end{aligned} \tag{6b}$$

This solution explicitly exhibits the periodicity of the fields in the "extended lab frame". (See Fig. 3) At any

fixed position, the time period is $2L/v$, while the length period is $2L$. At the instant when the images cover one another ($vt = nL$) the electric field vanishes while the magnetic field is minimum midplane between images and maximum in the image planes.

In Ea. (6) we have written,

$$\Omega_j \equiv \gamma k_j v$$

$$K_j \equiv \gamma k_j L$$

Since B does not vanish at $t = 0$, it follows that the solution constructed from images as outlined above, while satisfying boundary and singularity conditions does not satisfy the initial data that E = B = 0 for $t \leq 0$ everywhere in the box. In so far as the solution so constructed incorporates the presence of the source, it is a particular solution (viz. to the wave equations). The total solution to our initial value problem is obtained by adding to this particular integral a solution to the homogeneous wave equations. This final form gives E = B = 0 at $t = 0$, is singular on $z = vt$, E_{||} vanishes at the walls, and is causal. It follows that it is the correct Green's solution to the stated problem.

3b. The Homogeneous Component and Total Green's Function.

In so far as $B_z = 0$ in the inhomogeneous solution (6), it represents a TM wave. It follows that a superposition of TM waves must be added thereto to give the desired null effects. These are TM_{0jp} modes, j referring to wave number k_j and p to z -harmonic dependence. The zero relates to Bessel function order. The eigen frequencies ω_{jp} which accompany these modes are

$$\omega_{jp}^2 = c^2 [k_j^2 + (p\pi/L)^2]$$

Superposition of these modes give the fields,

$$\begin{aligned} \mathcal{E}_z &= \sum_p \sum_j \epsilon_{pj} J_0(k_j r) \cos\left(\frac{p\pi z}{L}\right) \exp(-i\omega_{pj} t) \\ \mathcal{E}_r &= \sum_p \sum_j \epsilon_{pj} \left(\frac{p\pi}{Lk_j}\right) J_1(k_j r) \sin\left(\frac{p\pi z}{L}\right) \exp(-i\omega_{pj} t) \\ \mathcal{B}_\phi &= \sum_p \sum_j \epsilon_{pj} \left(\frac{i\omega_{pj}}{ck_j}\right) J_1(k_j r) \cos\left(\frac{p\pi z}{L}\right) \exp(-i\omega_{pj} t) \end{aligned} \quad (7)$$

The coefficient ϵ_{pj} is to be determined. Both the real and imaginary components of these fields, respectively, are solutions to the homogeneous wave equations. At $t = 0$

$\text{Im } \mathcal{B}_\phi \equiv B_\phi^h$ (h denotes homogeneous) appears as,

$$B_{\phi}^h = - \sum_p \sum_j \epsilon_{pj} \left(\frac{\omega_{pj}}{ck_j} \right) J_1(k_j r) \cos \left(\frac{p\pi z}{L} \right)$$

Comparison with Eq. (6) indicates that ϵ_{pj} must be chosen so that

$$\frac{4q\gamma\beta}{J_1^2 r_0^2} \frac{\cosh K_j (1 - \frac{z}{L})}{\sinh K_j} = \sum_p \epsilon_{pj} \frac{\omega_{pj}}{ck_j} \cos \frac{p\pi z}{L}$$

To solve for ϵ_{pj} we employ the Fourier decomposition,

$$\frac{\cosh K(1 - \frac{z}{L})}{K \sinh K} = \sum_{p=-\infty}^{\infty} \frac{\cos \frac{p\pi z}{L}}{(\pi p)^2 + K^2}$$

To validate this representation we rewrite the summation as an integral in the complex p -plane.

$$\sum_{p=-\infty}^{\infty} \frac{\cosh (\pi p z / L)}{(\pi p)^2 + K^2} = \frac{1}{2\pi i} \int_C \frac{\cos (\pi p z / L) \cot \pi p}{[(\pi p)^2 + K^2]} dp \equiv I$$

The curve "C" encircles the real p -axis as shown in Fig. 4a.

Using the cos addition law we obtain,

$$I = \frac{1}{2\pi i} \int_C \frac{\cos \pi p (1 - \frac{z}{L}) dp}{[(\pi p)^2 + K^2] \sin \pi p} - \frac{1}{2\pi i} \int_C \frac{\sin (\pi p z / L) dp}{[(\pi p)^2 + K^2]}$$

Since the second integrand is an analytic function along the whole real axis, its integral vanishes. To evaluate the first integral, we distort the contour "C" into "C₁" and "C₂" as shown in Fig. 4b. This gives the two residues from the poles at $\pi p = \pm iK$ which add to yield the desired result.

For ϵ_{pj} we then have,

$$\frac{\omega_{pj} \epsilon_{pj}}{ck_j} = \frac{4q\gamma\beta K_j}{J_1^2 r_o^2 [(\pi p)^2 + K_j^2]}$$

where $-\infty \leq p \leq +\infty$.

Substituting this value for ϵ_{pj} into Eq. (7), taking the Im part thereof and adding the resultant fields to the inhomogeneous solution, Eq. (6), gives

$$\begin{aligned} E_z &= \frac{4q}{r_o^2} \sum_{j=0}^{\infty} \frac{J_0(k_j r)}{J_1^2} \left\{ \frac{\sinh \Omega_j t \cosh K_j (1 - \frac{z}{L})}{\sinh K_j} \right. \\ &\quad \left. - \sum_{p=-\infty}^{\infty} \frac{K_j \Omega_j \sin \omega_{pj} t \cos \frac{\pi p z}{L}}{\omega_{pj} [(\pi p)^2 + K_j^2]} \right\} \\ &\equiv \frac{4q}{r_o^2} \sum_j \frac{J_0(k_j r)}{J_1^2} U_j^>(z, t, v, L) \end{aligned} \quad (8a)$$

$$E_r = \frac{4q\gamma}{r_0} \sum_{j=1}^{\infty} \frac{J_1(k_j r)}{J_1^2} \left\{ \frac{\sinh \Omega_j t \sinh K_j (1 - \frac{z}{L})}{\sinh K_j} \right. \quad (8a)$$

cont'd.

$$- \sum_{p=-\infty}^{+\infty} \frac{\pi p \Omega_j \sin \omega_{pj} t \sin \frac{\pi p z}{L}}{\omega_{pj} [(\pi p)^2 + K_j^2]} \left. \right\}$$

$$\equiv \frac{4q\gamma}{r_0} \sum_j \frac{J_1(k_j r)}{J_1^2} V_j > (z, t, v, L)$$

$$B_\phi = \frac{4q\gamma\beta}{r_0} \sum_{j=0}^{\infty} \frac{J_1(k_j r)}{J_1^2} \left\{ \frac{\cosh \Omega_j t \cosh K_j (1 - \frac{z}{L})}{\sinh K_j} \right.$$

$$- \sum_{p=-\infty}^{\infty} \frac{K_j \cos \omega_{pj} t \cos \frac{\pi p z}{L}}{[(\pi p)^2 + K_j^2]} \left. \right\}$$

$$\equiv \frac{4q\gamma\beta}{r_0} \sum_j \frac{J_1(k_j r)}{J_1^2} W_j > (z, t, v, L)$$

The second identification in each case above serves to define the (z, t) dependent forms, U , V , and W . These are the fields for $z > vt$.

For $z < vt$, merely substitute the inhomogeneous terms in the above expressions with those given in Eq. (6b). The homogeneous component of the total solution (for $z \leq L$) is then continuous across the plane $z = vt$ while the inhomogeneous

component includes the singularity at the particle position.

For $z < vt$ there results,

$$\begin{aligned}
 E_z &= \frac{4q}{r_0^2} \sum_j \frac{J_0(k_j r)}{J_1^2} \left\{ \frac{\sinh K_j \frac{z}{L} \sinh (K_j - \Omega_j t)}{\sinh K_j} \right. \\
 &+ \left. \sum_p \frac{K_j \Omega_j \sin \omega_{pj} t \cos \frac{\pi p z}{L}}{\omega_{pj} [(\pi p)^2 + K_j^2]} \right\} \\
 &\equiv - \frac{4q}{r_0^2} \sum_j \frac{J_0(k_j r)}{J_1^2} U^<
 \end{aligned}$$

$$\begin{aligned}
 E_r &= \frac{4q\gamma}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \left\{ \frac{\sinh K_j \frac{z}{L} \sinh (K_j - \Omega_j t)}{\sinh K_j} \right. \\
 &- \left. \sum_p \frac{\pi p \omega_j \sin \omega_{pj} t \sin \frac{\pi p z}{L}}{\omega_{pj} [(\pi p)^2 + K_j^2]} \right\} \tag{8b} \\
 &\equiv \frac{4q\gamma}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} V^<
 \end{aligned}$$

$$\begin{aligned}
 B_\phi &= \frac{4q\gamma\beta}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \left\{ \frac{\cosh K_j \frac{z}{L} \cosh (K_j - \Omega_j t)}{\sinh K_j} \right. \\
 &- \left. \sum_p \frac{K_j \cos \omega_{pj} t \cos \frac{\pi p z}{L}}{[(\pi p)^2 + K_j^2]} \right\} \\
 &\equiv \frac{4q\gamma\beta}{r_0^2} \sum_j \frac{J_1(k_j r)}{J_1^2} W^<
 \end{aligned}$$

Eqs. (8) with $q = 1$ are the components of the relativistic Green's tensor field $G_{\mu\nu}(\underline{x}, t)$ (this notation will be used below). The components of G are the values of the fields at \underline{x}, t due to a point unit charge moving with v along the axis of a cylindrical grounded box, which was at $z = 0$ at $t = 0$.

3c. Causality

To show that the components of G as given by Eq. (8) are all causal the summation over p is converted to a contour integration in the p -plane (after multiplying by $\cot \pi p$). The following addition formulas are used.

$$\cos \pi p \left(1 - \frac{z}{L}\right) = \cos \frac{\pi p z}{L} \cos \pi p + \sin \frac{\pi p z}{L} \sin \pi p$$

$$\sin \pi p \left(1 - \frac{z}{L}\right) = - \sin \frac{\pi p z}{L} \cos \pi p + \cos \frac{\pi p z}{L} \sin \pi p$$

In each case it is found that U, W, V , vanish for $z > ct$. For $z \leq ct$, in the general case a simplifying contour distortion is not evident and we must work with the summations over p in their generic form.

To illustrate the causal property of the solution (8) we consider B_ϕ . We wish to show that

$$W = 0, \quad z > ct$$

or equivalently (deleting the j index)

$$\frac{\cosh \Omega t \cosh K(1-\frac{z}{L})}{K \sinh K} = \sum_{p=-\infty}^{+\infty} \frac{\cos \omega_p t \cos \frac{\pi p z}{L}}{(\pi p)^2 + K^2}$$

$$= \frac{1}{2\pi i} \int_C dp \frac{\cos \omega_p t \cos \frac{\pi p z}{L} \cot \pi p}{[(\pi p)^2 + K^2]} \equiv \Lambda$$

where the contour C is depicted in Fig. 4a. From the cos law the latter integral is decomposed into,

$$\Lambda = \frac{1}{2\pi i} \int_C \frac{\cos \omega_p t \cot \pi p (1-\frac{z}{L}) dp}{(\pi p)^2 + K^2}$$

$$- \frac{1}{2\pi i} \int_C \frac{\cos \omega_p t \sin \frac{\pi p z}{L}}{(\pi p)^2 + K^2}$$

The second integrand is regular in the domain enclosed by C so that only the first integral contributes to Λ . Expanding the integrand of this first integral gives

$$\Lambda = \frac{1}{4\pi i} \int_C \frac{\cos \left[\omega_p t + \pi p(1-\frac{z}{L}) \right] + \cos \left[\omega_p t - \pi p(1-\frac{z}{L}) \right]}{[(\pi p)^2 + K^2] \sin \pi p} dp$$

$$\equiv \frac{1}{4\pi i} \int_C \tilde{\Lambda} dp$$

In the limit that $\text{Im } p \equiv p' \rightarrow \pm\infty$, $\omega_p \rightarrow c\pi p'/L$ and

$$\left| \left[(\pi p)^2 + K^2 \right] \tilde{\Lambda} \right| \rightarrow 2 \left| e^{-|\pi p|} \left\{ \exp \left[-\frac{\pi p'}{L} (z - ct) + \pi p' \right] \right. \right. \\ \left. \left. + \exp \left[\frac{\pi p'}{L} (z - ct) - \pi p' \right] + \exp \left[\frac{\pi p'}{L} (z + ct) - \pi p' \right] \right. \right. \\ \left. \left. + \exp \left[-\frac{\pi p'}{L} (z + ct) + \pi p' \right] \right\} \right|$$

For $p' \rightarrow \pm\infty$ the RHS of this latter expression goes to zero providing $ct < z$ and $2L > z + ct$. Both inequalities are satisfied for $ct < z < L$. It follows that for these values of z and t the curve C may be distorted into C_1 and C_2 as depicted in Fig. 4b. This picks up the two residues at $K = \pm i\pi p$ to yield,

$$\Lambda = \frac{1}{2\pi i} \int_C \frac{\cos \omega_p t \cos \pi p (1 - \frac{z}{L})}{\left[(\pi p)^2 + K^2 \right] \sinh \pi p} = \frac{1}{2\pi i} \left[\int_{C_1} + \int_{C_2} \right] \\ = \frac{\cosh \Omega t \cosh K (1 - \frac{z}{L})}{K \sinh K}$$

It follows that $W = 0$ for $z > ct$. Similar constructions hold for U and V .

3d. The Semi-Infinite and Completely Infinite Pipe.

The solution to the above problem as given by Eq. (8) reduces to a very simple form in the interval $t < L/c$. In this interval the pulse is not influenced by the forward wall at $z = L$ and must reduce to that due to a moving charged particle in a semi-infinite tube with walls at $z = 0$ and $z = \infty$.

In this limit ($L \rightarrow \infty$), the inhomogeneous contributions in U , V and W become

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}_i \rightarrow e^{-\gamma k z} \begin{pmatrix} \sinh \Omega t \\ \sinh \Omega t \\ \cosh \Omega t \end{pmatrix}$$

In the homogeneous p -summations, the discrete variable p becomes the continuous variable η through the transformation,

$$\frac{\pi p}{L} \rightarrow \eta \quad ; \quad \frac{\pi}{L} \rightarrow d\eta$$

There results,

$$\frac{\pi v}{\Omega} \begin{pmatrix} U \\ V \\ W \end{pmatrix}_h \rightarrow \begin{pmatrix} \Omega \int_{-\infty}^{\infty} \frac{d\eta \sin \omega t \cos \eta z}{\omega(\eta^2 + \gamma^2 k^2)} \\ v \int_{-\infty}^{\infty} \frac{\eta d\eta \sin \omega t \sin \eta z}{\omega(\eta^2 + \gamma^2 k^2)} \\ \int_{-\infty}^{\infty} \frac{d\eta \cos \omega t \cos \eta z}{\eta^2 + \gamma^2 k^2} \end{pmatrix}$$

Rewriting the trigonometric products in exponential form, and closing the integration along the real p-axis with the upper or lower semi-circle, depending on which of these the integrand vanishes, gives the following

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}_h = - \begin{pmatrix} U \\ V \\ W \end{pmatrix}_i \quad z > ct$$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}_h = 0 \quad z \leq ct$$

Only the inhomogeneous solution survives in the limit $L \rightarrow \infty$, and we obtain for $vt < z \leq ct$. (ahead of the particle)

$$E_z = \frac{4q}{r_0^2} \sum \frac{J_0(k_j r)}{J_1^2} e^{-\gamma k_j z} \sinh \Omega_j t$$

$$E_r = \frac{4q}{r_0^2} \sum \frac{J_1(k_j r)}{J_1^2} e^{-\gamma k_j z} \sinh \Omega_j t \quad (9a)$$

$$B_\phi = \frac{4q\gamma\beta}{r_0^2} \sum \frac{J_1(k_j r)}{J_1^2} e^{-\gamma k_j z} \cosh \Omega_j t$$

For $z > ct$ the fields vanish. Behind the particle ($0 \leq z < vt$) we find,

4. Applications

4a. The Wake Fields.

At $vt = L$, the particle leaves the cylindrical box. To obtain the fields in the box for $vt > L$, we do the following. The fields "behind" the particle (for $vt \leq L$) are given by Eqs. (8b). Symbolically we write these fields as,

$$G = G^i + G^h$$

Where, as before, i denotes inhomogeneous and h denotes homogeneous. Let us construct a field G , which satisfies the following initial value problem:

$$\square^2 \tilde{G} = 0 \quad \text{for } vt > L$$

and

$$\tilde{G}(r, vt, z) = G^i(r, vt, z) \quad \text{at } vt = L$$

It then follows that

$$\square^2 [\tilde{G} + G^h] = 0 \quad \text{for } vt > L$$

(since G^h is a solution to the homogeneous equation for all time) and

coefficients ϵ_{pj} which enter follow from the above Fourier decomposition.

$$-\frac{\omega_{pj} \epsilon_{pj}}{ck_j} = \frac{4q\gamma\beta(-)^b K_j}{J_1^2 r_o^2 [(\pi p)^2 + K_j^2]}$$

Combining these fields with G^h (the second terms in each of Eq. (8a)) gives the closed fields in the wake domain,

$$E_z = \frac{4q}{r_o^2} \sum_j \sum_p \frac{K_j \Omega_j \cos \frac{\pi p z}{L}}{\omega_{pj} [(\pi p)^2 + K_j^2]} \frac{J_0(k_j r)}{J_1^2} \left[-\sin \omega_{pj} t + (-)^b \sin \omega_{pj} \left(t - \frac{L}{v}\right) \right]$$

$$E_r = \frac{4q\gamma}{r_o^2} \sum_j \sum_p \frac{\pi p \Omega_j \sin \frac{\pi p z}{L}}{\omega_{pj} [(\pi p)^2 + K_j^2]} \frac{J_1(k_j r)}{J_1^2} \left[-\sin \omega_{pj} t + (-)^p \sin \omega_{pj} \left(t - \frac{L}{v}\right) \right] \quad (11)$$

$$B_\phi = \frac{4q}{r_o^2} \sum_j \sum_p \frac{K_j \cos \frac{\pi p z}{L}}{[(\pi p)^2 + K_j^2]} \frac{J_1(k_j r)}{J_1^2} \left[-\cos \omega_{pj} t + (-)^p \cos \omega_{pj} \left(t - \frac{L}{v}\right) \right]$$

$$\equiv \frac{4q\gamma\beta}{r_o^2} \sum_j \frac{J_1(k_j r)}{J_1^2} \bar{W}_j$$

$$U < 16L \left(\frac{\alpha\gamma\beta}{r_0} \right)^2 \sum_p \sum_j \frac{K_j^2}{J_1^2 \left[(\pi p)^2 + K_j^2 \right]^2}$$

The p-summation may be evaluated as follows. Define

$$M \equiv \sum_{p=-\infty}^{+\infty} \frac{1}{\left[(\pi p)^2 + K^2 \right]^2} = \frac{1}{2\pi i} \int_C \frac{\cot \pi p}{\left[(\pi p)^2 + K^2 \right]^2}$$

where the contour C is depicted in Fig. 4. Since

$$\lim_{p \rightarrow \pm i\infty} |\cot \pi p| = 1$$

the contour C may be distorted into C_1 and C_2 to yield,

$$M = \frac{2\pi}{(2K)^2 \sinh^2 K} \left[1 + \frac{\sinh K \cosh K}{K} \right]$$

It follows that,

$$U < \gamma\pi L \left(\frac{\alpha\gamma\beta}{r_0} \right)^2 \sum_j \frac{1}{J_1^2 \sinh K_j} \left[1 + \frac{\sinh K_j \cosh K_j}{K_j} \right]$$

In the limit as $j \rightarrow \infty$ the second term in the summand goes to a constant and gives a divergent sum. This singularity

(evident from Eq. (12)) stems from idealization that the point particle q leaves (as well as enters) through a point hole. When the particle reaches the far wall it coalesces with its (nearest) image and stops. The resulting singular pulse is trapped in the box. This conclusion is consistent with Ott's⁸ calculation for the transition radiation problem⁹ in which a particle is incident on a grounded plane. After the particle passes through the plane a hemispherical wave propagates away from the wall carrying zero-field behind it and the previous field in front of it. The fields at the wave surface are singular. In the similar problems with a hole in the plane^{10,11} this singularity is obviated. Similarly, in the problem considered herein, if the series above, written in the form,

$$U < \sum_j U_j$$

is cut-off at $r_0 k_j \approx \frac{2\pi r_0}{d}$ then the sum is finite. This would be, roughly, the energy deposited in a finite cylindrical box with holes of diameter d in its end plates.

The large order zeros of J_0 go as $r_0 k_j \approx \pi j$ so that there are

$$j \sim 2r_0/d$$

terms in the cut-off series of Eq. (12). It follows that an upper estimate of this series is given by

$$U < j U_{j_{\max}} = j \frac{4\pi\gamma q^2 \beta^2}{r_0} = \frac{8\pi\gamma q^2 \beta^2}{d}$$

Our assumption that the particle does not lose too much of its energy to the stimulated wake fields will be valid if

$$\frac{U}{(\gamma-1)m_0 c^2} \ll 1$$

For an electron with $\beta \approx 1$ one obtains

$$r_0 \gg d \gg 10^{-13} \text{ cm}$$

which is easily satisfied in most practical cases. The left inequality insures that the hole is, at most, a small perturbation in the included analysis.

4c. The Charged Line Segment

In this section we consider a line charge of length $b < L$ and charge q . There are four relevant epochs. (See Fig. 5) In epoch (1), the segment is partially in the cavity, ($vt < b$). In epoch (2), the segment is completely in the cavity, ($b \leq vt \leq L$).

In epoch (3), the segment is leaving the cavity, ($L - b < vt - b < L$).
 In the fourth epoch, the pulse has completely left the cavity,
 ($vt - b \leq L$).

In the first two epochs we only need the Green's fields given by Eq. (8). We recall that these fields (with $q = 1$) are those at (\underline{x}, t) due to a point charge which entered the cavity at $t = 0$. In epoch (1), one obtains

$$F_{\mu\nu}(\underline{x}, t) = \frac{qv}{b} \int_0^t G_{\mu\nu}(\underline{x}, \tau) d\tau \quad t \leq b/v$$

More explicitly, the B field is given by:

$$B^{(1)} = \sum_j \frac{J_1(k_j r)}{J_1^2} \int_0^t W_j^> d\tau \quad vt < z \quad (15a)$$

ahead of the leading edge of the pulse, and

$$B^{(1)} = \sum_j \frac{J_1(k_j r)}{J_1^2} \left\{ \int_0^{T_z} W_j^> d\tau + \int_{T_z}^0 W_j^> d\tau \right\} \quad z < vt \quad (15b)$$

behind the leading edge of the pulse. The factors W are given in Eq. (8), while B and T_z are defined through,

$$B_\phi \equiv \frac{4qv\gamma\beta}{br_0^2} B$$

$$T_z \equiv z/v$$

The wake factor $\bar{W}(\eta, t)$ is defined by Eq. (11). It gives the wake field at "t" due to a particle which penetrated the far wall of the cavity at $\eta \equiv L/v$.

In the domain behind the after edge of the pulse

$$B^{(3)} = \sum_j \frac{J_1(k_j r)}{J_1^2} \left\{ \int_{t-T}^{T_L} W_j < d\tau + \int_{T_L}^t \bar{W}_j(\eta, t) d\eta \right\} \quad z < vt - b \quad (17b)$$

In the fourth epoch, $vt - b \geq L$, the solution is a superposition of homogeneous wake fields.

$$B^{(4)} = \sum_j \frac{J_1(k_j r)}{J_1^2} \int_{T_L}^{T_L + T_b} \bar{W}_j(\eta, t) d\eta \quad (18)$$

In addition to the trigonometric behavior, the time dependence of these fields includes the hyperbolic components,

$$\int \begin{pmatrix} \cosh \Omega \tau \\ \sinh \Omega \tau \end{pmatrix} \Omega d\tau = \begin{pmatrix} \sinh \Omega \tau \\ \cosh \Omega \tau \end{pmatrix}$$

For the third epoch, we recall that the time dependence of \bar{W} is given by the factor

$$\Lambda_{\bar{W}} = -\cos \omega t + (-)^P \cos \omega(t-\eta)$$

The integrals over Λ_1 which enter the above expressions for B are,

$$\int_{T_L}^t \Lambda_{\bar{W}} d\eta = (T_L - t) \cos \omega t + (-)^{P_{\omega}-1} \sin \omega(t-T_L)$$

which reveals a secular behavior. In the fourth epoch the modes are all harmonic. In general, all the time integrations in Eqs. (11-15) are simply performed.

In the extremely relativistic limit much of the segment may enter the cavity before the fields are influenced by the far wall. (See Fig. 7) The fields are then most simply obtained by integrating the asymptotic forms, Eq. (9). Writing only the B fields, we have for $vt < z < ct$ (and $vt < b$), in front of the leading edge of the pulse,

$$B_{\phi} = \frac{4qyv\beta}{br_o^2} \sum_j \frac{J_1(k_j r)}{J_1^2} e^{-\gamma k_j z} \frac{\sinh \Omega_j t}{\Omega_j}$$

and behind the leading edge of the pulse,

$$B_{\phi} = \frac{4qyv\beta}{br_o^2} \sum_j \frac{J_1(k_j r)}{J_1^2 \Omega_j} \left[1 - e^{-\Omega_j t} \cosh \gamma k_j z \right]$$

For any of these pulse problems if the total charge of the segment is q and I_A is beam current in amps, then

$$q = \frac{10}{c} I_A T_b \text{ esu}$$

4d. Return Current-Point and Line Charge Segment.

The surface current at the wall of the perfectly conducting cylinder is given by,

$$S_z = \frac{c}{4\pi} B_\phi(r_0) \quad (\text{stat amps/cm})$$

From Eq. (8) one obtains, for the propagating point charge,

$$S_z^{\gtrsim} = \frac{c}{\pi} \frac{q\gamma\beta}{r_0^2} \sum_j \frac{W_j^{\gtrsim}(z,t;v,L)}{J_1(k_j r_0)}$$

where, as before \gtrsim denotes $z \gtrsim vt$.

After the particle leaves the box, the surface current is obtained from the wake solution, Eq. (11). For the line pulse of charge, there are four distinct epochs ($\alpha = 1, \dots, 4$). The surface current during these intervals is given by.

$$S_z^{(\alpha)} = \frac{q\gamma c^2 \beta^2}{\pi b r_0^2} B^{(\alpha)}$$

with $B^{(\alpha)}$ given in Eqs. (15-18). The time behavior of S_z follows B so that we expect hyperbolic dependence in the first two epochs with secular dependence entering in the third epoch, during which time the segment is leaving the cavity.

5. Conclusions

In this analysis we have studied the fields induced in a finite, closed, cylindrical cavity, with perfectly conducting walls, by a relativistically moving charged point particle. Referring to Fig. 6, we see that the fields conveniently divide into four distinct domains. In the region bounded by the triangle OAB the fields are those of a point charge moving in a semi infinite tube. All events in this domain are not influenced by the forward wall (whose world-line is $z = L$). In the domain bounded by the triangle OAD all fields vanish since all events in this domain are not influenced by the particle. Above the line $ct = -z + 2L$, but outside the triangle OAB, the full solution, Eq. (8) comes into play. The wake fields, Eq. (11) come into play above the line $t = T_L$, again, excluding the points in OAB.

In calculating the energy in the wake fields a singularity enters owing to the idealization in our model that the point charge enters and leaves the cavity through point holes. This in turn necessitates that the point charge coalesce with its image and stop in zero time. The infinite deceleration launches a singular pulse back into the cavity. Any finite hole obviates this singularity. For an electron, it was found that a hole of diameter exceeding one fermi insures that the initial energy of the particle is large compared to the energy excited in the wake fields.

The point charge solution was used to obtain the fields of a line segment of charge. Here it was found that the time domain relevant to the problem divides into four epochs as depicted in Fig. 5. The space-time diagram for this problem is shown in Fig. 7. In the triangle OAB the fields are those of a finite segment propagating down a semi-infinite cylinder. Above the line $ct = -z + 2L$ the total solution as given by Eqs. (15-18) comes into play.

The formalism introduced herein for obtaining the relativistic Green's solution may be easily extended to a variety of problems, provided one is able to formulate the static solution in terms of images. Such problems include motion in a cylinder of arbitrary cross section; the motion of any charge configuration which lies in a plane of constant z ; motion in a dielectric medium. The limitation of the theory is that it does not account for interaction between charges in a given configuration.

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Discussions on these topics with Prof. E. Ott proved invaluable. My sincere appreciation and gratitude are extended to him. Special thanks are also due Profs. N. Rostoker, P. McIsaac and Dr. J. Hieronymus for their very worthwhile suggestions relating to this paper.

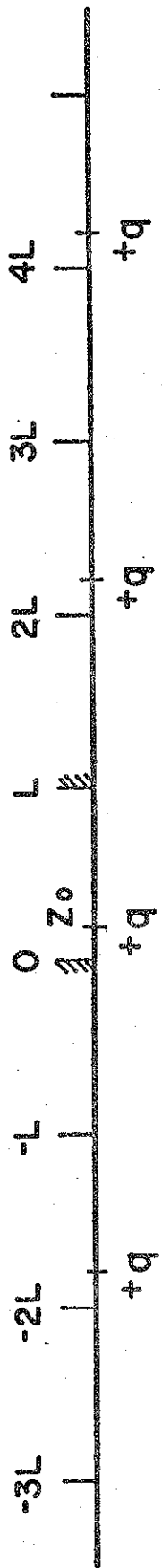
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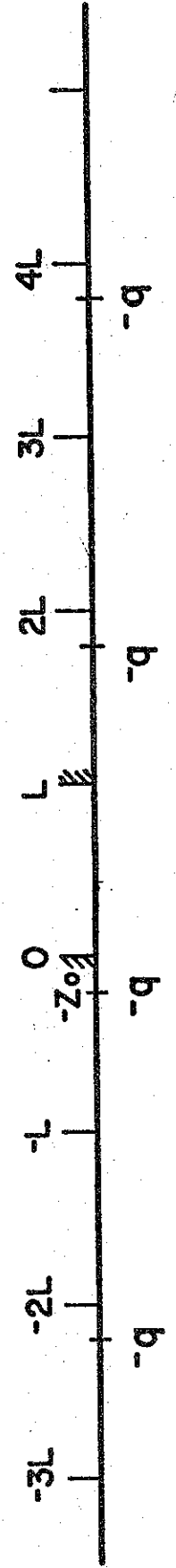
List of Figure Captions

1. Image locations for a particle in a cylindrical box.
2. Positive and negative image frames.
3. The extended lab frame.
4. The contours C , C_1 , and C_2 .
5. Relevant time intervals for the line charge problem.
6. Space-time diagram for the point charge.
7. Space-time diagram for the line-charge.



$$Z_n^{(+)} = 2nL + Z$$

$$n = -\infty \dots +\infty$$

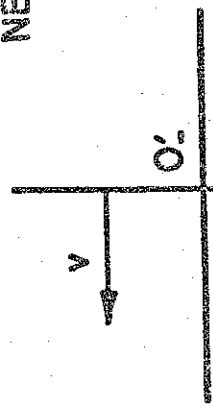


$$Z_n^{(-)} = 2nL - Z$$

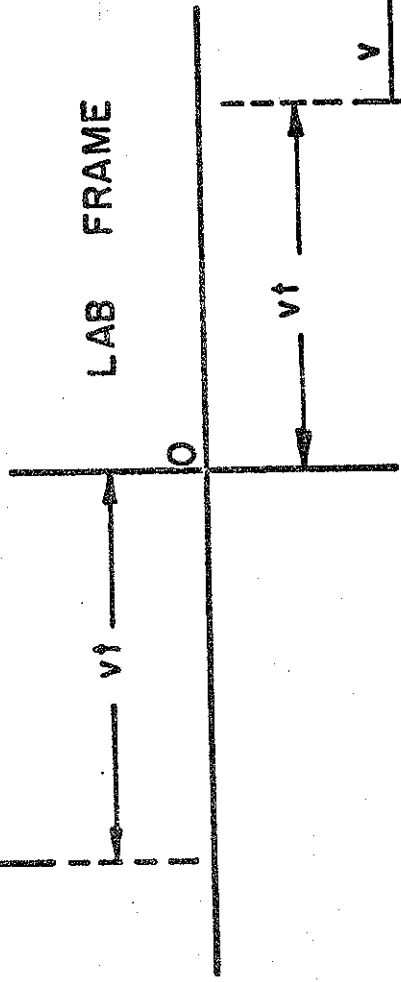
$$n = -\infty \dots +\infty$$

FIG. 1

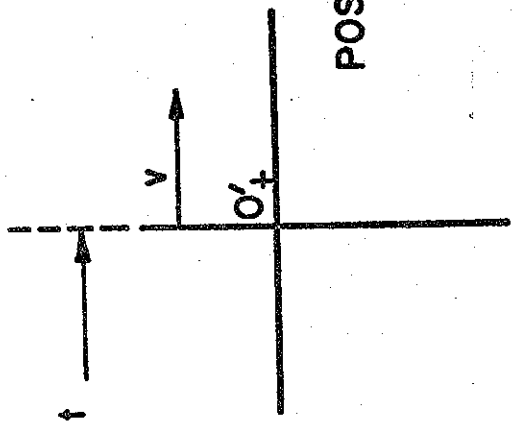
NEGATIVE CHAIN FRAME



LAB FRAME



POSITIVE CHAIN FRAME



$$Z'_+ = \gamma(Z - vt)$$

$$Z'_- = \gamma(Z + vt)$$

FIG 2

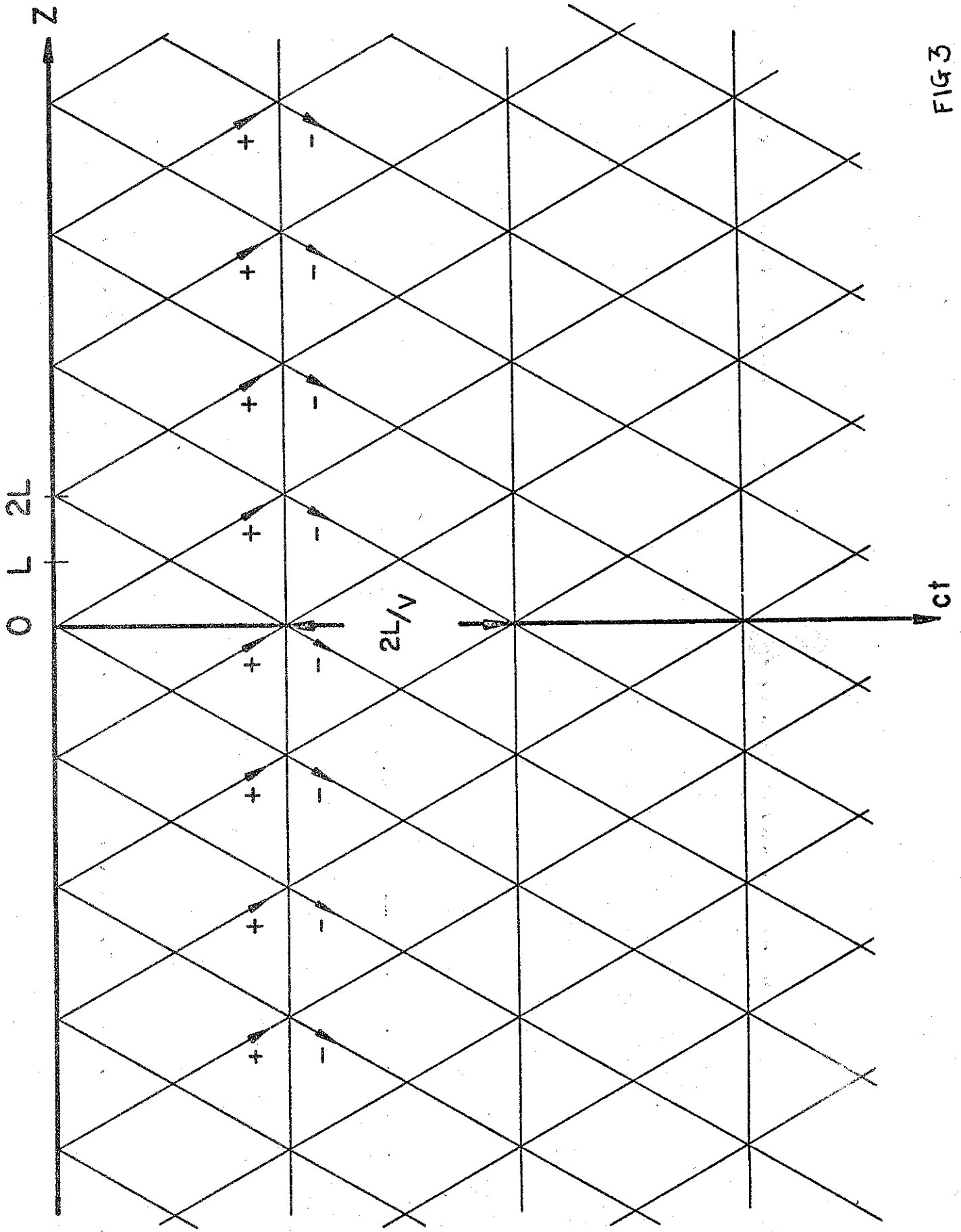


FIG 3

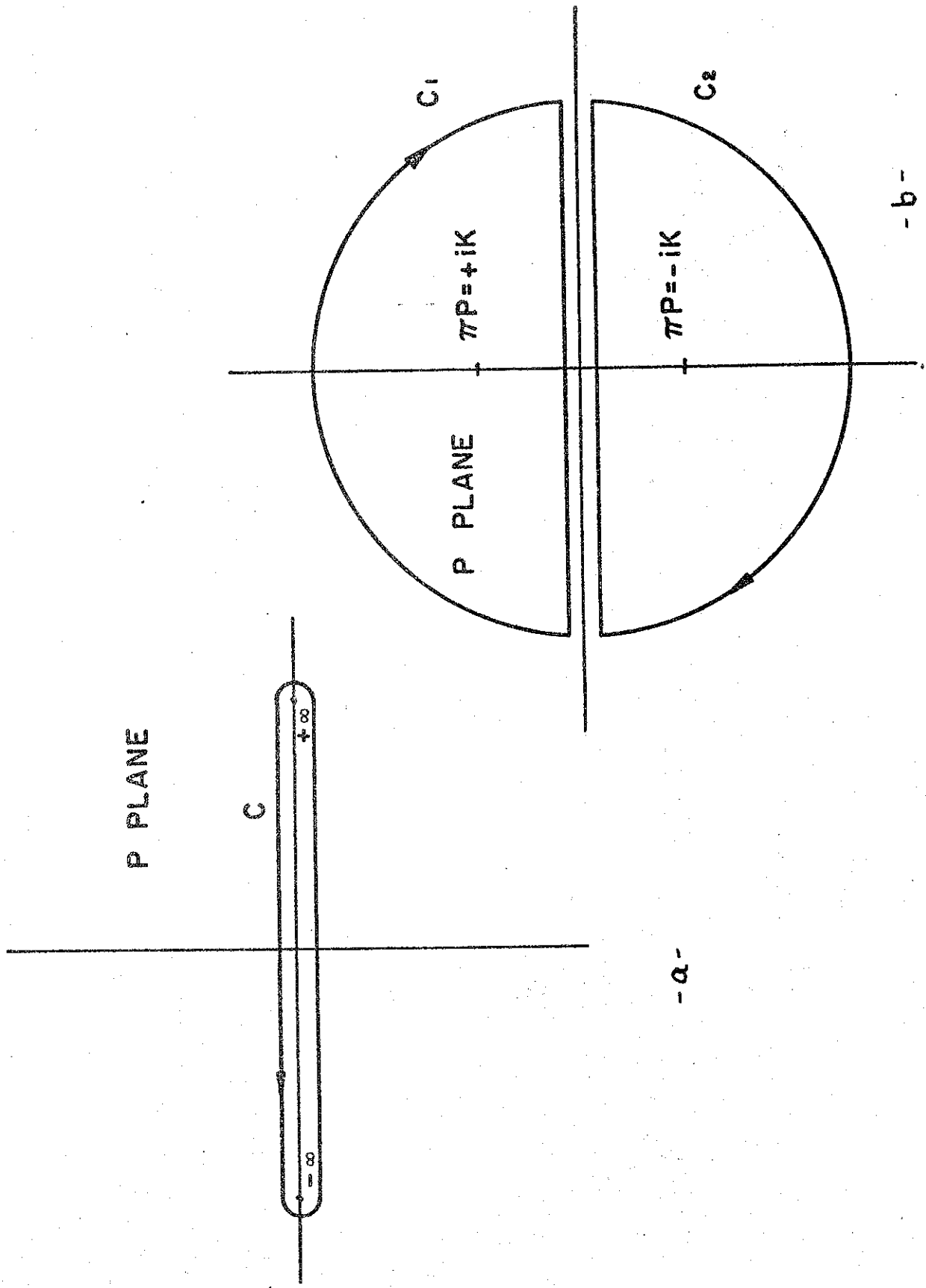
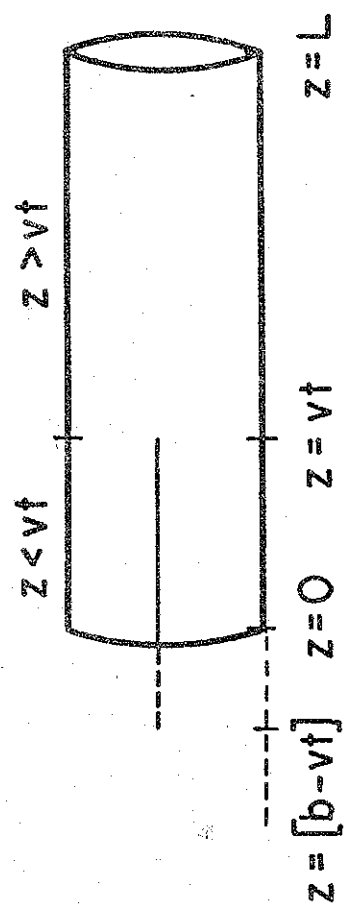
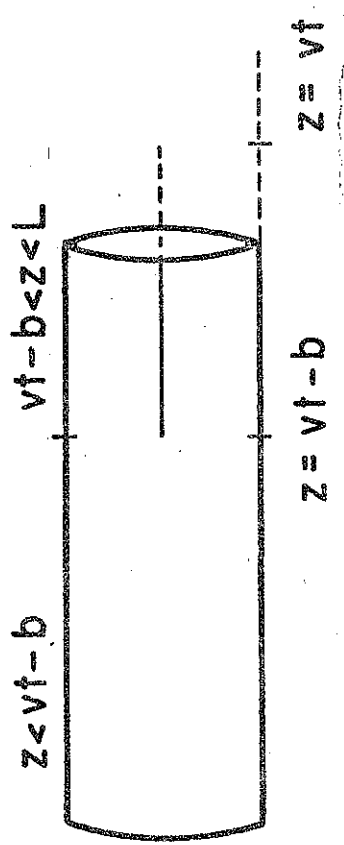


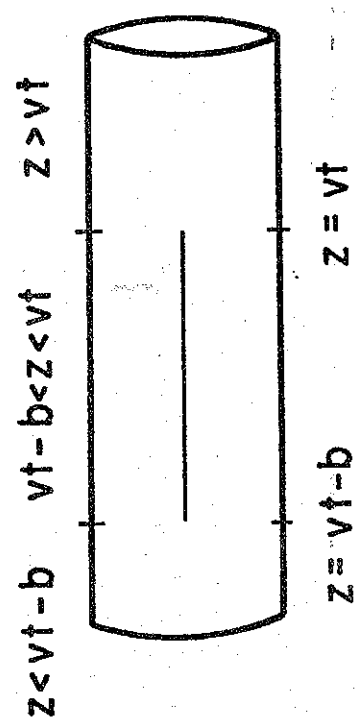
FIG 4



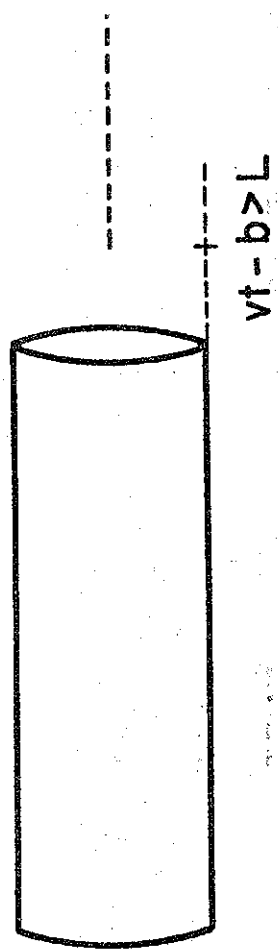
EPOCH 1



EPOCH 3



EPOCH 2



EPOCH 4

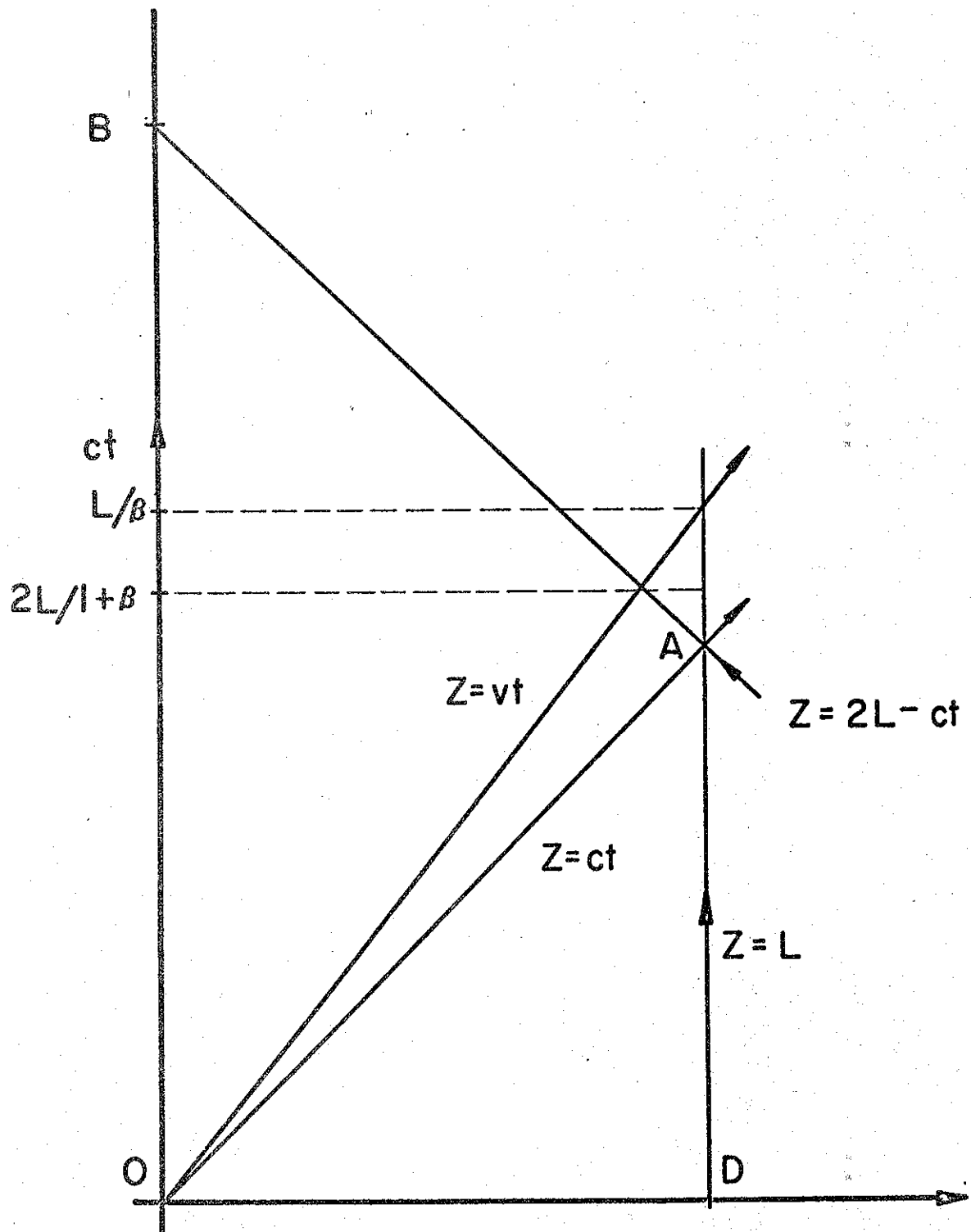


FIG. 6